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NEAR ISOMETRIES IN THE CLASS OF L^1-PREDUALS[†]

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ABSTRACT

We construct preduals of l_1 which are nearly isometric without being isometric. We also show that if X is nearly isometric to a $C(K)$ space with K first countable, then they are in fact isometric.

1. Introduction

For a pair X, Y of Banach spaces, define their distance coefficient by $d(X, Y) = \inf\{\|T\|\|T^{-1}\|\}$ where the infimum is taken over all isomorphisms T from X onto Y. The spaces X and Y are called nearly isometric if $d(X, Y) = 1$. Examples of nearly-isometric but non-isometric spaces are well known (probably the first example of this type was constructed in an unpublished paper by A. Pelczynski in 1960).

This paper investigates the possibility of the occurence of such a phenomenon in the class of L^1-preduals, and was motivated by the uniqueness problem of the Gurari space(s). Gurari [4] has constructed an L^1-predual with special extension properties (see also [5], [9] for other constructions and the importance of this space in the general theory of L^1-preduals). It was noted by Gurari that this space is unique up to near-isometry, but it is still an open question whether it is unique up to isometry.

A theorem of Amir [1] and Cambern [2] states that if $d(C(K), C(H)) < 2$ then K is homeomorphic to H and thus $C(K)$ is isometric to $C(H)$. It might have been thought that the L^1-preduals behave in a manner similar to their prototype, the $C(K)$ spaces. This is however false and Section 2 is devoted to a construction of nearly-isometric non-isometric preduals of l_1. We also show that a similar construction can be made in various classes of L^1-preduals.

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In Section 3, we give a positive result and show that if $d(X, Y) = 1$ and X is a $C(K)$ space (or more generally, a $C_\sigma(K)$ space), and if also K is first countable then X is actually isometric to Y.

In Section 4 we examine briefly the other classical spaces, the $L^p(\mu)$ spaces. We consider only the separable case and show that in this case there is a constant K_p such that $d(L^p(\mu), L^p(\nu)) < K_p$ implies that $L^p(\mu)$ is isometric to $L^p(\nu)$. Let us recall in this connection that it is known (cf. [6]) that if $X = L^p(\mu)$ for some $1 \leqq p < \infty$ and some measure μ (or if $X^* = L^1(\mu)$) and if Y is such that $d(X, Y) = 1$ then $Y = L^p(\nu)$ for some measure ν and the same p (respectively, $Y^* = L^1(\nu)$).

For a Banach space X we denote by B_X its closed unit ball. If K is a convex subset of X we denote by $E(K)$ the set of extreme points of K. For a set K in a dual space X^* we denote its ω^*-closure by $\bar{K}$. We shall deal with real Banach spaces only. The results are also valid in the complex case although some modifications are needed in Section 3.

2.

For every $0 < a < 1$ let X_a be the space of all convergent sequences $y = (y_n)$ of reals such that $\lim y_n = ay_1$ equipped with the sup norm. It is easy to see that X_a^* is isometric to l_1 and that

$$\bar{E}(B_{X_a^*}) = \{\pm e_n : n = 1,2,\cdots\} \cup \{\pm ae_1\}$$

where e_n is the evaluation functional $e_n(y) = y_n$.

For a countable subset $A = \{a_i\}$ of $(0,1)$ let $X_A = (\Sigma \oplus X_{a_i})_{c_0}$. Again X_A^* is isometric to l_1, and if we identify $X_{a_i}^*$ in the natural way as a subspace of X_A^* we get that $\bar{E}(B_{X_A^*}) = \cup \bar{E}(B_{X_{a_i}^*}) \cup \{0\}$.

PROPOSITION. *Let $A = \{a_i\}$ and $B = \{b_i\}$ be two different countable dense subsets of $(0,1)$. Then $d(X_A, X_B) = 1$, but X_A is not isometric to X_B.*

PROOF. For $0 < a \leqq b < 1$ define $T : X_a \to X_b$ by

$$(Ty)_n = \begin{cases} \dfrac{a}{b}\, y_1 & n = 1 \\ y_n & n > 1. \end{cases}$$

Then T is an isomorphism onto with $\|T\|\,\|T^{-1}\| = b/a$ and thus $d(X_a, X_b) \leqq b/a$. Since X_A and X_B do not depend on the order of the a_i's and b_i's, we can assume, using the density of A and B that given any $\epsilon > 0$, we have $(1+\epsilon)^{-1} < b_i/a_i < 1+\epsilon$ for every i. By the previous remark this implies that

$d(X_{a_i}, X_{b_i}) \leqq 1+\epsilon$ for every i and thus also $d(X_A, X_B) \leqq 1+\epsilon$. Since this is true for every $\epsilon > 0$ we get that X_A is nearly isometric to X_B.

Assume now that there is an isometry T from X_A onto X_B. Then T^* is a ω^*-continuous isometry of X_B^* onto X_A^*, and in particular maps $\bar{E}(B_{X_B^*})$ isometrically onto $\bar{E}(B_{X_A^*})$. But by the remarks in the beginning of this section we obtain that

$$\{\|x\| : x \in \bar{E}(B_{X_B^*})\} = B \cup \{0\} \cup \{1\}$$

$$\{\|x\| : x \in \bar{E}(B_{X_A^*})\} = A \cup \{0\} \cup \{1\}$$

and since $A \neq B$ these sets are different, a contradiction.

REMARK. It is interesting to check in what classes of L^1-preduals [7] one could construct examples of this type. The spaces X_A constructed above are M spaces. A similar example can be constructed in the class of $A(K)$ spaces as well: For $0 < a < 1$ let Y_a be the space of all convergent sequences $y = (y_n)$ such that $\lim y_n = ay_1 + (1-a)y_2$. For a countable subset $A = \{a_i\}$ of $(\frac{1}{2}, 1)$ let $Y_A = (\Sigma \bigoplus Y_{a_i})_{c_0}$. The space Y_A can be identified cannonically as a subspace of $C(K)$ where K is the set of ordinals $1 \leqq \xi \leqq \omega^2$ (with the usual order topology): We identify Y_A with

$$\{f \in C(K) : f(\omega^2) = 0 \quad \text{and} \quad \lim_{n\to\infty} f(\omega \cdot i + n) = a_i f(\omega \cdot i + 1) + (1-a_i) f(\omega \cdot i + 2)\}.$$

We put $Z_A = \operatorname{sp}\{Y_A, 1\}$ where 1 is the function which is identically 1 on K. Then Z_A^* is isometric to l_1 and its unit ball has an extreme point, and thus it is an $A(K)$ space (see [5]). Again one can show that if A and B are dense in $(\frac{1}{2}, 1)$ then $d(Z_A, Z_B) = 1$ but they are not isometric unless $A = B$.

3.

Since $d(X_a, c) \leqq a^{-1}$ it is clear that there is no constant $\lambda > 1$ so that $d(C(K), X) \leqq \lambda$ and $X^* = L^1(\mu)$ will already ensure that $C(K)$ is isometric to X, like in the Amir-Cambern theorem. The following theorem is, however, true (see [7] for the definition of $C_\sigma(K)$ spaces):

THEOREM 1. *If $d(X, Y) = 1$ and Y is a $C_\sigma(K)$ space with K first countable, then X is isometric to Y.*

We need first some notation and simple lemmas.

We identify $C_\sigma^*(K)$ with the space of all measures on K which are anti-symmetric with respect to σ. We denote by φ_k the element of $C_\sigma(K)^*$

defined by $\varphi_k(f) = f(k) = -f(\sigma k)$. It is well known that if $X = C_\sigma(K)$, then $E(B_{X^*}) = \{\varphi_k : k \in K, \varphi_k \neq 0\}$ and that if $E(B_{X^*})$ is not closed then $\bar{E}(B_{X^*}) = E(B_{X^*}) \cup \{0\}$.

From now on we assume that K is first countable.

LEMMA 1. *Let $\epsilon > 0$ and let $\varphi_k = \int \nu_s d\mu(s)$ be a ω^*-representation in $C_\sigma^*(K)$, where $\varphi_k \neq 0$ and μ is a probability measure on $\{\nu \in C_\sigma^*(K) : \|\nu\| \leqq 1 + \epsilon\}$. Then there exists a Baire set E which is a subset of*

$$\{\nu : \|\nu - \varphi_k\| \leqq 2\epsilon,\ \|\nu\| \leqq 1 + \epsilon\},\ \textit{with}\ \mu(E) \geqq 1/3.$$

PROOF. Choose $\{f_n\}$ in $C_\sigma(K)$ such that $\|f_n\| = 1 = f_n(k)$ and such that f_n converge pointwise to $\chi_{\{k\}} - \chi_{\{\sigma k\}}$. Define

$$E_n = \{\nu\,;\, \nu(f_n) \geqq 1 - \epsilon/2, \|\nu\| \leqq 1 + \epsilon\}; \quad E = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n.$$

Clearly E is a Baire subset of $\{\nu : \|\nu - \varphi_k\| \leqq 2\epsilon, \|\nu\| \leqq 1 + \epsilon\}$ and it suffices to show that for all n, $\mu(E_n) \geqq 1/3$. Indeed,

$$1 = f_n(k) = \int \left(\int f_n(t) d\nu_s(t) \right) d\mu(s) \leqq \mu(E_n)(1 + \epsilon) + \mu(K \setminus E_n)(1 - \epsilon/2)$$
$$= 1 + \epsilon(\mu(E_n) - \tfrac{1}{2}\mu(K \setminus E_n))$$

which implies that $\mu(E_n) \geqq 1/3$.

LEMMA 2. *Let $\epsilon > 0$ and let T be an isomorphism from $C_\sigma(K)$ onto X such that $\|f\| \leqq \|Tf\| \leqq (1 + \epsilon)\|f\|$. Then for every $k \in K$ such that $\varphi_k \neq 0$ there exists an $x^*(k) \in E(B_{X^*})$ with $\|x^*(k) - T^{*^{-1}}\varphi_k\| \leqq 2\epsilon$.*

PROOF. By the Choquet representation theorem [8], there exists a probability measure μ on B_{X^*} which vanishes on every Baire set disjoint from $E(B_{X^*})$ such that $T^{*^{-1}}\varphi_k = \int_{B_{X^*}} x^* d\mu(x^*)$ and thus $\varphi_k = \int_{B_{X^*}} T^* x^* d\mu(x^*)$. Since $\|T^* x^*\| \leqq 1 + \epsilon$ we can apply Lemma 1. The set E is a Baire set with $\mu(E) \geqq 1/3$ and thus contains an extreme point $x^*(k)$. Clearly $\|x^*(k) - T^{*^{-1}}\varphi_k\| \leqq 2\epsilon$.

The preceding lemma was true for general X, but in the special case where $X^* = L^1(\mu)$ more can be said. In this case the distance between any two extreme points in B_{X^*} is 2, and thus we obtain that $x^*(k)$ is uniquely determined provided $\epsilon < 1/2$. We shall denote by ψ the map $k \to x^*(k)$ and call it the map induced by T.

As was remarked earlier if $X = C_\tau(H)$ then the only accumulation point of $E(B_{X^*})$ is (possibly) the origin. This implies easily that in this case the induced map ψ is ω^*-continuous.

COROLLARY. *Let $T: C_\sigma(K) \to C_\tau(H)$ be an isomorphism onto such that $\|T\| \|T^{-1}\| \leqq 1 + \epsilon$ with $\epsilon < 1/3$. Then there exists an isometry $W : C_\sigma(K) \to C_\tau(H)$ such that $\|T - W\| \leqq 2\epsilon(1 + \epsilon)$.*

PROOF. Let ψ be the map induced by T and ψ_1 the map induced by T^{-1}. By applying T^* to the inequality $\|T^{*-1}\varphi_k - \psi(k)\| \leqq 2\epsilon$ we get that $\|T^*\psi(k) - \varphi_k\| \leqq 2\epsilon(1 + \epsilon) < 1$. By the uniqueness of the extreme point which is closer than 1 to $T^*\psi(k)$ we get that $\varphi_k = \psi_1(\psi(k))$. Thus ψ, which is ω^*-continuous by the previous remark is a homeomorphism of $\bar{E}(B_{C_\sigma^*(K)})$ onto $\bar{E}(B_{C_\tau^*(H)})$ which satisfies $\psi(z^*) = -\psi(z^*)$ and thus induces an isometry W of $C_\sigma(K)$ onto $C_\tau(H)$ by defining for $f \in C_\sigma(K)$, $Wf(h) = \psi^{-1}(\varphi_h)(f)$. Clearly $W^{*-1}\varphi_k = \psi(k)$ and $\|W - T\| \leqq 2\epsilon(1 + \epsilon)$.

The preceding corollary is a weak generalization of the Amir-Cambern theorem to $C_\sigma(K)$ spaces. We would like to point out however (without going into detail) that by using the techniques of Cambern [2] one can get the full generalization and show that $d(C_\sigma(K), C_\tau(H)) < 2$ implies that $C_\sigma(K)$ is isometric to $C_\tau(H)$.

LEMMA 3. *Let $X^* = L^1(\mu)$ and let T_1, T_2 be isomorphisms of $C_\sigma(K)$ onto X such that $\|T_1\| \|T_1^{-1}\| \|T_2\| \|T_2^{-1}\| < 1 + \epsilon$ which $\epsilon < 1/6$. Then there exists an isometry W of $C_\sigma(K)$ onto itself such that T_1 and $T_2 W$ induce the same map ψ.*

PROOF. Denote by ψ the map induced by T_1. The operator $T_2^{-1}T_1 : C_\sigma(K) \to C_\sigma(K)$ satisfies the conditions of the corollary and thus induces an isometry W of $C_\sigma(K)$ onto itself such that $\|W^{*-1}\varphi_k - (T_2^{-1}T_1)^{*-1}\varphi_k\| \leqq 2\epsilon$ for every $k \in K$. Thus

$$\|(T_2 W)^{*-1}\varphi_k - \psi(k)\| \leqq \|T_2^{*-1} W^{*-1}\varphi_k - T_2^{*-1}(T_2^{-1}T_1)^{*-1}\varphi_k\|$$

$$+ \|T_1^{*-1}\varphi_k - \psi(k)\| \leqq \|T_2^{*-1}\| 2\epsilon + 2\epsilon < 1.$$

By the uniqueness of the extreme point closer than 1 to $(T_2 W)^{*-1}\varphi_k$ we get that ψ is indeed the map induced by $T_2 W$.

PROOF OF THEOREM 1. By [6], X^* is an $L^1(\mu)$ space. Let $T_n : C_\sigma(K) \to X$ be a sequence of isomorphisms such that $\|T_n\| \|T_n^{-1}\| \leqq 1 + 10^{-n}$ and let ψ be the map induced by T_1. By Lemma 3 one can find other isomorphisms S_n such that $\|S_n\| \|S_n^{-1}\| \leqq 1 + 10^{-n}$ and such that they all induce the same map ψ. Thus for every $x \in X$ we have that

$$|(S_k^{-1} - S_m^{-1})x(k)| = |(S_n^{*-1} - S_m^{*-1})\varphi_k(x)| \leqq 2(10^{-n} + 10^{-m}) \|x\|$$

and the operators S_n^{-1} form a Cauchy sequence, in the operator norm. Clearly the limit of the sequence $\{S_n^{-1}\}_{n=1}^\infty$ is an isometry from X onto $C_\sigma(K)$.

4.

In this section we examine the stability of the $L^p(\mu)$ spaces under small isomorphism. The case $p = 1$ follows by duality from the Amir-Cambern theorem.

THEOREM 2. *If $d(L^1(\mu), L^1(\nu)) < 2$ then $L^1(\mu)$ is isometric to $L^1(\nu)$.*

PROOF. By passing to the dual we also get that $d(L^1(\mu)^*, L^1(\nu)^*) < 2$, but every L^∞ space is isometric to a $C(K)$ space and thus, by the theorem of Amir-Cambern, $L^\infty(\mu)$ is isometric to $L^\infty(\nu)$. By a theorem of Grothendieck [3] this implies that $L^1(\mu)$ is isometric to $L^1(\nu)$.

For $p > 1$ we shall deal with the separable case only. As is well known, every separable L^p space $(p \neq 2)$. is isometric to one of the following: $L^p(0,1)$, l_p, $(L^p(0,1) \oplus l_p^n)_p$, $1 \leqq n \leqq \aleph_0$. As $L^p(0,1)$ is not isomorphic to a subspace of l_p we get the complete answer by the following theorem.

THEOREM 3. *For every $p \neq 2$ there exists an $\epsilon = \epsilon(p)$ such that if $d(L^p(0,1) \oplus l_p^n, L^p(0,1) \oplus l_p^m) < 1 + \epsilon$ then $m = n$.*

PROOF. The cases $p = \infty$ and $p = 1$ were already dealt with, and by duality we can assume that $2 > p > 1$. Assume that $n > m$ and let $T: L^p(0,1) \oplus l_p^n \to L^p(0,1) \oplus l_p^m$ be an isomorphism with $\|x\| \leqq \|Tx\| \leqq (1+\epsilon)\|x\|$. Denote by $\{e_i\}_{i=1}^n$ the unit vectors in l_p^n and by $\{f_j\}_{j=1}^m$ the unit vectors in l_p^m.

It is easy to check that for every i and for every $g \in L^p(0,1) \oplus l_p^n$ we have that $\max\|e_i \pm g\|^p \geqq 1 + \|g\|^p$. Thus, by putting $h = Tg$ we get

$$(1) \qquad \max \| Te_i \pm h \|^p \geqq (1+\epsilon)^{-p}(1 + \| h \|^p) \text{ for every } h \in L^p(0,1) \oplus l_p^m.$$

Fix now i and let $Te_i = \varphi + \sum_{j=1}^m a_j f_j$ where $\varphi \in L^p(0,1)$. We claim that if ϵ is small enough there exists k such that $|a_k|^p > 1 - 2^{p-2}$. Clearly if $i_1 \neq i_2$ then $k_1 \neq k_2$ (provided ϵ is small enough) which is impossible if $m < n$.

Assume that $\max|a_j|^p \leqq 1 - 2^{p-2}$. In this case we could find a subset J of $\{1, \cdots, m\}$ such that $|\sum_{j \in J}|a_j|^p - \sum_{j \notin J}|a_j|^p| \leqq 1 - 2^{p-2}$. We can also find a subset A of $[0,1]$ such that $\int_A |\varphi|^p = \int_{[0,1] \setminus A} |\varphi|^p$. Define now $h = (2\chi_A(t) - 1)\varphi + \sum_{j \in J} a_j f_j - \sum_{j \notin J} a_j f_j$. An easy computation shows that

$$\max \|Te_i \pm h\|^p \leqq 2^{p-1}(\|Te_i\|^p + 1 - 2^{p-2}) \leqq 2^{p-1}((1+\epsilon)^p + 1 - 2^{p-2})$$

$$\text{and} \quad \|h\| = \|Te_i\| \geqq 1.$$

Substituting this h in (1) we get that $2^{p-1}((1+\epsilon)^p + 1 - 2^{p-2}) \geqq 2(1+\epsilon)^{-p}$ which is a contradiction provided ϵ is small enough. This proves Theorem 3.

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